8a) If $G = \langle S \rangle$ then the homomorphism ϕ is completely determined by $\{\phi(s) : s \in S\}$. In other words, if $\tilde{\phi} : G \to H$ is any homomorphism with $\tilde{\phi}(s) = \phi(s)$, $\forall s \in S$, then $\tilde{\phi} = \phi$.

Pf: Suppose $\tilde{\phi} : G \to H$ is a hom. with $\tilde{\phi}(s) = \phi(s)$, $\forall s \in S$.

Then $\forall g \in G$, $\exists n \in \mathbb{N}$, $g_1, \dots, g_n \in S$, and $u_1, \dots, u_n \in \{\pm 1\}$ s.t. $g = g_1^{u_1} g_2^{u_2} \cdots g_n^{u_n}$. Then

$$\tilde{\phi}(g) = \tilde{\phi}(g_1)^{u_1} \tilde{\phi}(g_2)^{u_2} \cdots \tilde{\phi}(g_n)^{u_n}$$

$$= \phi(g_1)^{u_1} \phi(g_2)^{u_2} \cdots \phi(g_n)^{u_n} = \phi(g). \quad \blacksquare$$

Important: When defining a hom. $\phi : G \to H$, just choosing values for $\phi(s)$, for $s \in S$, doesn't guarantee that ϕ will actually extend to a hom. on all of G. (see examples below)

Exs:

1) Find all homomorphisms $\phi : \mathbb{Z}/8\mathbb{Z} \to C_6$.

Write $C_6 = \langle x \rangle = \{e, x, x^2, x^3, x^4, x^5\}$.

Since $\mathbb{Z}/8\mathbb{Z} = \langle 1 \rangle$, any hom. $\phi : \mathbb{Z}/8\mathbb{Z} \to C_6$ is determined $\phi(1)$. There are six choices to consider:

- If $\phi(1) = e$ then for ϕ to be a hom. we must have,

$\forall\, 0 \le k \le 7$, $\phi(k) = \phi(\underbrace{1+1+\cdots+1}_{k\text{-times}}) = \underbrace{\phi(1)\cdot\phi(1)\cdots\phi(1)}_{k\text{-times}} = e^k = e$.

This does define a hom., the trivial hom.

- Can't have $\phi(1) = x$:
  - If ϕ is a hom. then $|\overset{=x}{\phi(1)}| \mid |1|$. (property 5a)

    But $|1| = 8$, $|x| = 6$, and $6 \nmid 8$.
  - Another way to see this: If ϕ were a hom.

    w/ $\phi(1) = x$ then $\phi(k) = \phi(1)^k = x^k$, $\forall k \in \mathbb{Z}$.

    But $1 \equiv 9 \bmod 8$, so $\phi(9) = \phi(1) = x \neq \underset{= x^3}{x^9}$.
- Similarly, can't have $\phi(1) = x^2, x^4,$ or x^5

  $|x^2| = \frac{6}{(2,6)} = 3$, $\quad |x^4| = \frac{6}{(4,6)} = 3$, $\quad |x^5| = \frac{6}{(5,6)} = 6$.

  (video on cyclic groups)
- If ϕ were a hom. with $\phi(1) = x^3$ then it would force

  $\phi(k) = \phi(1)^k = x^{3k}$, $\quad 0 \le k \le 7$:

k	0	1	2	3	4	5	6	7
$\phi(k)$	e	x^3	e	x^3	e	x^3	e	x^3

  This does define a (non-trivial) hom.

So, there are 2 homs. from $\mathbb{Z}/8\mathbb{Z}$ to C_6: the trivial hom., and the hom. determined by the rule $1 \mapsto x^3$.

2) Find all homs. $\phi: S_3 \to C_6$.

order 1   order 6

Write $S_3 = \langle (1\,2), (1\,2\,3) \rangle$, $\quad C_6 = \langle x \rangle = \{e, x, x^2, x^3, x^4, x^5\}$.

order 2   order 3   order 3   order 2

Any hom. $\phi: S_3 \to C_6$ is determined by $\phi((1\,2))$ and $\phi((1\,2\,3))$

Since $|\phi(g)| \mid |g|$, must have:

$\phi((1\,2)) = e$ or x^3 and $\phi((1\,2\,3)) = e, x^2,$ or x^4. (6 possibilities)

- Can't have $|\phi((1\,2))| = 2$ and $|\phi((1\,2\,3))| = 3$:

  Since $\phi(S_3) \leq C_6$, by Lagrange's thm. $|\phi(S_3)| = 1, 2, 3,$ or 6.

  If $|\phi(S_3)| = 6$ then ϕ would be an isomorphism. However,

  C_6 is Abelian and S_3 is not, so this can't happen,

  which means that $|\phi(S_3)| = 1, 2,$ or 3.

  Finally, $\forall g \in S_3$, we have $|\phi(g)| \,\big|\, |\phi(S_3)|$.

  Therefore we can't have $|\phi((1\,2))| = 2$ and $|\phi((1\,2\,3))| = 3$.

- Suppose $\phi((1\,2)) = \phi((1\,2\,3)) = e$.

  Then ϕ extends to the trivial hom. from S_3 to C_6.

- Suppose $\phi((12)) = x^3$, $\phi((123)) = e$.

If ϕ extends to a hom. then it must be given by:

σ	e	$(1\,2\,3)$	$(1\,3\,2)$	$(1\,2)$	$(2\,3)$	$(1\,3)$
$\phi(\sigma)$	e	e	e	x^3	x^3	x^3

Scratch work: $(1\,2\,3)^2 = (1\,3\,2) \Rightarrow \phi((1\,3\,2)) = \phi((1\,2\,3))^2 = e^2 = e$

$(12)(1\,2\,3) = (2\,3) \Rightarrow \phi((2\,3)) = \phi((12))\,\phi((1\,2\,3)) = x^3 \cdot e = x^3$

$(12)(1\,2\,3)^2 = (1\,2)(1\,3\,2) = (1\,3) \Rightarrow \phi((1\,3)) = \phi((12))\,\phi((1\,2\,3))^2 = x^3 \cdot e^2 = x^3$

Note: This map can also be defined by

$$\phi(\sigma) = \begin{cases} x^3 & \text{if } \sigma \text{ is odd}, \\ e & \text{if } \sigma \text{ is even}. \end{cases}$$

From this, it is easy to see that it is a homomorphism.

- Suppose $\phi((12)) = e$, $\phi((123)) = x^2$.

If ϕ extends to a hom. then it must be given by:

σ	e	$(1\,2\,3)$	$(1\,3\,2)$	$(1\,2)$	$(2\,3)$	$(1\,3)$
$\phi(\sigma)$	e	x^2	x^4	e	x^2	x^4

Scratch work: $\phi((1\,3\,2)) = \phi((1\,2\,3))^2 = x^4$

$\phi((2\,3)) = \phi((12))\,\phi((1\,2\,3)) = x^2$

$\phi((1\,3)) = \phi((12))\,\phi((1\,2\,3))^2 = x^4$

Problem: $(23)^2 = e$, so $e = \phi((23)^2) \neq \phi((23))^2 = x^4$.

So this choice does not extend to a hom.

- Suppose $\phi((1\,2)) = e$, $\phi((1\,2\,3)) = x^4$.

If ϕ extends to a hom. then it must be given by:

σ	e	$(1\,2\,3)$	$(1\,3\,2)$	$(1\,2)$	$(2\,3)$	$(1\,3)$
$\phi(\sigma)$	e	x^4	x^2	e	x^4	x^2

But $e = \phi((2\,3)^2) \neq \phi((2\,3))^2 = x^8 = x^2$.

So this choice does not extend to a hom.

So, there are 2 homs. from S_3 to C_6: the trivial hom., and the hom. determined by the rule $(1\,2) \mapsto x^3$, $(1\,2\,3) \mapsto e$.

8b) Suppose that $G = \langle S \rangle$, that $\phi : S \to H$, and that any relation satisfied by elements $s_1, \dots, s_n \in S$ is also satisfied with these replaced by $\phi(s_1), \dots, \phi(s_n)$. Then ϕ extends uniquely to a homomorphism from G to H.

Note: When a presentation for G is given (i.e. using generators and relations), you only need to check the above condition for the relations in the presentation.

3) $D_6 \cong S_3$:

$D_6 = \langle r, s \mid r^3 = s^2 = e,\ rs = sr^{-1} \rangle$

Let $\phi(r) = (1\ 2\ 3)$, $\phi(s) = (1\ 2)$. Then:

- $r^3 = e$ and $(1\ 2\ 3)^3 = e$
- $s^2 = e$ and $(1\ 2)^2 = e$
- $rs = sr^{-1}$ and $\underbrace{(1\ 2\ 3)(1\ 2)}_{(1\ 3)} = \underbrace{(1\ 2)\overbrace{(1\ 2\ 3)^{-1}}^{(3\ 2\ 1)}}_{(1\ 3)}$

Therefore, ϕ extends to a hom. $\phi : D_6 \to S_3$.

But $3 = |\phi(r)| \mid |\phi(D_6)|$ and $2 = |\phi(s)| \mid |\phi(D_6)|$

$\Rightarrow |\phi(D_6)| = 6 \Rightarrow \phi$ is bijective, therefore it is an isomorphism.